Simple but High-Accuracy Approximations for $n!$

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Abstract

Computing $n!$ efficiently and accurately is notoriously difficult. Many have proposed approximations, varying in accuracy and in computational complexity. Interesting and useful approximations are both accurate and computationally inexpensive, and, if possible, exact up to machine-precision floating-point numbers. In this paper, we exploit an observation on a previous approximation by Hodgman to obtain a new class of correction terms using simple, but optimal given their degree, rational functions. We show that the proposed approximations are more accurate than some of the best-known approximations while remaining computationally inexpensive.

Keywords: Approximation, Asymptotic Error, Factorial, Rational Function.

1 An approximation by Hodgman

The best-known series to compute $n!$ is most certainly Stirling's [18] series

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right), \tag{1}$$

where the numerators and denominators are given by sequences A001163 and A001164 from the OEIS [16, 17]. We may truncate the series to

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n}\right) \tag{2}$$

as a trade-off between accuracy and speed of computation. However, using only $1 + \frac{1}{12n}$ as a correction term makes eq. (2) underestimate $n!$. To counter this effect, a proposition, likely by Hodgman [6, p. 326], is to use

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - 1}\right), \tag{3}$$

but it now overestimates $n!$—while yielding a larger error than eq. (2), which may explain why this approximation disappeared from ulterior editions. But if eq. (2) underestimates $n!$, eq. (3) overestimates $n!$, and $\frac{1}{12n-\alpha}$ is a continuous function, then, by the intermediate value theorem, there must be a value or function α, with $-1 < \alpha < 0$, such that the error is zero. However, to obtain a computationally simple formula, we must limit the form α can take. For this work, we will limit ourselves to rational functions of small degree d in n, noted $\alpha(d, n)$. Let then

$$P_{\alpha,d}(n) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n + \alpha(d, n)}\right) \tag{4}$$

be the approximation to $n!$ in which $\alpha(d, n)$ is used.

We will find the optimal rational function $\alpha(d, n)$, where the degree d is chosen in order to satisfy one's trade-off between accuracy and computational cost. For $d = 0$, $\alpha(0, n)$ will be a constant, while for $d > 0$, $\alpha(d, n)$ will be a non-constant rational function of degree d. In either case, it will yield a correction term of degree $d + 1$, as we will show.

In this paper, we will present a new method to find the optimal rational function $\alpha(d, n)$ for any desired degree d, starting, in section 2, with the special case $d = 0$, then, in section 3, for arbitrary degree d. In section 4, we will compare our results with some of the previously known approximations. Section 5 discusses the implementation and computational complexity of our proposed approximations. We conclude in section 6.

2 Optimal constant α

To find the optimal expression for $\alpha(d, n)$, we first notice that the squared error

$$E = \Big(n! - P_{\alpha,d}(n)\Big)^2$$

is convex in $\alpha(d, n)$. We can therefore find $\alpha(d, n)$ by solving $\dfrac{\partial E}{\partial \alpha(d, n)} = 0$ for $\alpha(d, n)$. This is equivalent to solving

$$1 + \frac{1}{12n + \alpha(d, n)} = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots \tag{5}$$

for $\alpha(d, n)$. Using α as a shorthand for $\alpha(d, n)$, we rework eq. (5) to obtain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2} + \frac{571\alpha}{207360n^3} - \frac{163879}{1451520n^3} + \cdots \tag{6}$$

If we want α to be a constant, that is, use $\alpha(0, n)$, we keep only the constant terms from eq. (6), and we find $\alpha = -\frac{1}{2}$. Substituting $\alpha = -\frac{1}{2}$ back into eq. (4), we obtain the optimal asymptotic approximation (for α constant)

$$n! \sim P_{\alpha,0}(n) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2}}\right), \tag{7}$$

resulting in a first degree rational function for the correction term,

$$n! \sim P_{\alpha,0}(n) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{24n + 1}{24n - 1}\right).$$

3 Optimal Degree-Limited Solutions

To solve for $\alpha(d,n)$ for a given degree d, we must retain from eq. (6) only terms which are of degree d or less. If we chose $d = 1$, we retain the terms of degree 0 (constants) and 1, that is,

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n},$$

from which we get, by isolating α, $\alpha(1,n) = -\frac{1}{2} + \frac{293}{720n+30}$. This yields the much better asymptotic approximation

$$\begin{aligned} n! \sim P_{\alpha,1}(n) &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{293}{720n+30}}\right) && \text{(8a)} \\ &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{360n+15}{4320n^2+139}\right), && \text{(8b)} \end{aligned} \quad (8)$$

resulting in a second degree rational function for the correction term. Solving eq. (6) for a second degree rational function $\alpha(2,n)$, we retain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2},$$

giving us $\alpha(2,n) = -\frac{1}{2} + \frac{7032n+293}{17280n^2+720n-556}$. The approximation becomes

$$\begin{aligned} n! \sim P_{\alpha,2}(n) &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{7032n+293}{17280n^2+720n-556}}\right) && \text{(9a)} \\ &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{17280n^2+720n-556}{207360n^3+571}\right), && \text{(9b)} \end{aligned} \quad (9)$$

which now has a third degree correction term. Solving eq. (6) for a third degree rational function $\alpha(3,n)$, we retain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2} + \frac{571\alpha}{207360n^3} - \frac{163879}{1451520n^3},$$

yielding $\alpha(3,n) = -\frac{1}{2} + \frac{1181376n^2+49224n-331755}{2903040n^3+120960n^2-93408n-7994}$, and therefore

$$\begin{aligned} n! \sim P_{\alpha,3}(n) &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{1181376n^2+49224n-331755}{2903040n^3+120960n^2-93408n-7994}}\right) && \text{(10a)} \\ &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1451520n^3+60480n^2-46704n-3997}{17418240n^4-163879}\right), && \text{(10b)} \end{aligned} \quad (10)$$

resulting in a fourth degree rational function for the correction term.

We could find $\alpha(4,n)$, $\alpha(5,n)$, etc., to yield increasingly more accurate approximations, but each new solution would be also increasingly computationally demanding. For a good trade-off between accuracy and computation, we are likely to retain only low-degree approximations, such as eqs. (7) to (10). Complexity of evaluation is discussed in section 5.

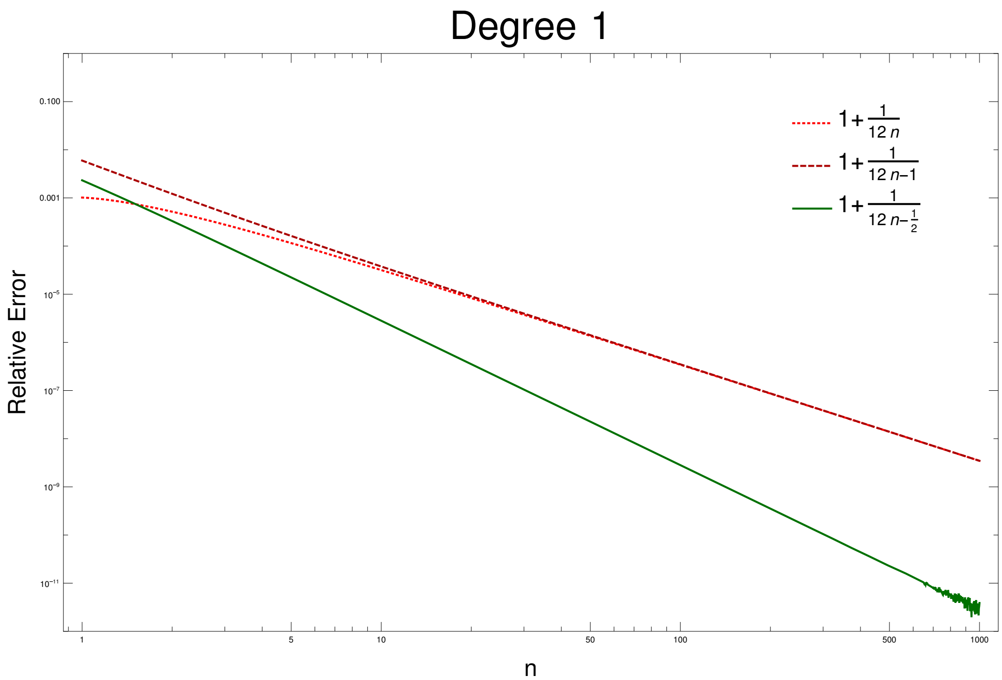

Figure 1: Comparisons of correction terms of degree 1.

4 Compared Accuracy

To compare the accuracy of the proposed approximations, we will use the absolute relative error

$$E(n!, a) = \left|\frac{n! - a}{n!}\right|,$$

where a is the approximation to $n!$ being evaluated. Since we are interested in simple approximations, we will exclude from this study costly approximations involving trigonometric functions, infinite series, (infinite) continued fractions, or Bernoulli numbers [20, 11, 8]. We will, however, compare our solutions to Mohanty's and Rummens' formula [9], which is a response to Weissman's [19] comment on Feller's and Kasper's approximation [4], itself derived from an identity by Burnside [2]. We will also compare our solutions to Nemes' [13], Mortici's [10], and Gosper's approximations [5].

In fig. 1 and in table 1, we compare the relative error of eqs. (2), (3), and (7) to $n!$ computed exactly using arbitrary-precision arithmetic. While Hodgman's correction, eq. (3), does worse than the Stirling series truncated to its first two terms, eq. (2), the added error is asymptotically negligible. However, the correction with $\alpha = -\frac{1}{2}$ does much better than either preceding approximations. We notice, in the lower-right corner of fig. 1, the effects of the approximation reaching machine-precision floating point accuracy. We will use arbitrary-precision arithmetic in subsequent figures.

We show the absolute relative errors of eqs. (7) to (10) in fig. 2 and table 2, using arbitrary-precision arithmetic. In fig. 2, one notices immediately dips in the absolute relative errors of eqs. (8) and (10). These correspond to values of n where the approximation *is* equal

n	$1+\frac{1}{12n}$	$1+\frac{1}{12n-1}$	$1+\frac{1}{12n-\frac{1}{2}}$
1	0.00102	0.00597	0.00232
10	0.00003	0.00004	2.81813×10^{-6}
20	8.30951×10^{-6}	9.05176×10^{-6}	3.53003×10^{-7}
30	3.74804×10^{-6}	3.96804×10^{-6}	1.04634×10^{-7}
40	2.12373×10^{-6}	2.21656×10^{-6}	4.41486×10^{-8}
50	1.36513×10^{-6}	1.41265×10^{-6}	2.26055×10^{-8}
60	9.50755×10^{-7}	9.78260×10^{-7}	1.30823×10^{-8}
70	6.99957×10^{-7}	7.17278×10^{-7}	8.23860×10^{-9}
80	5.36733×10^{-7}	5.48337×10^{-7}	5.51930×10^{-9}
90	4.24595×10^{-7}	4.32745×10^{-7}	3.87641×10^{-9}
100	3.44252×10^{-7}	3.50193×10^{-7}	2.82592×10^{-9}
500	1.38651×10^{-8}	1.39127×10^{-8}	2.26080×10^{-11}
1000	3.46925×10^{-9}	3.47519×10^{-9}	2.82600×10^{-12}
5000	1.38865×10^{-10}	1.38913×10^{-10}	2.26080×10^{-14}
10000	3.47193×10^{-11}	3.47252×10^{-11}	2.82600×10^{-15}

Table 1: Numerically compared errors for constant corrections: eq. (2), eq. (3) and eq. (7).

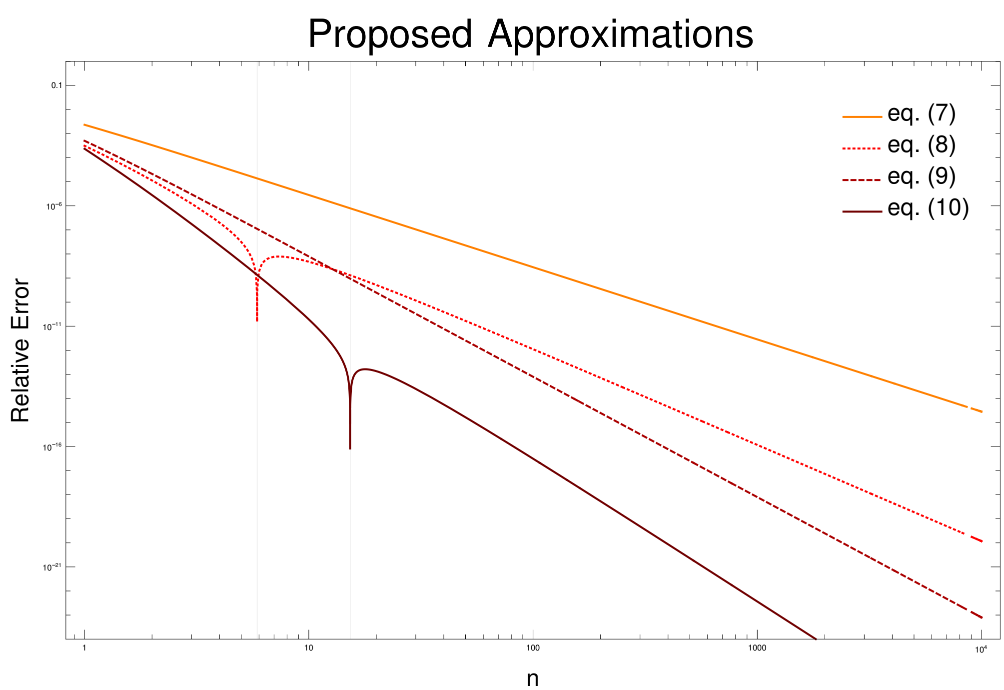

Figure 2: Proposed approximations and their absolute relative error to $n!$.

n	eq. (7)	eq. (8)	eq. (9)	eq. (10)
1	0.00232	0.00031	0.00050	0.00023
10	2.81813×10^{-6}	4.75012×10^{-9}	7.87352×10^{-9}	1.91489×10^{-11}
20	3.53003×10^{-7}	5.15163×10^{-10}	2.47478×10^{-10}	1.35261×10^{-13}
30	1.04634×10^{-7}	1.16269×10^{-10}	3.26261×10^{-11}	2.48111×10^{-14}
40	4.41486×10^{-8}	3.90880×10^{-11}	7.74545×10^{-12}	5.57148×10^{-15}
50	2.26055×10^{-8}	1.65760×10^{-11}	2.53852×10^{-12}	1.64257×10^{-15}
60	1.30823×10^{-8}	8.17576×10^{-12}	1.02028×10^{-12}	5.90763×10^{-16}
70	8.23860×10^{-9}	4.48322×10^{-12}	4.72080×10^{-13}	2.45804×10^{-16}
80	5.51930×10^{-9}	2.65882×10^{-12}	2.42145×10^{-13}	1.14196×10^{-16}
90	3.87641×10^{-9}	1.67487×10^{-12}	1.34377×10^{-13}	5.78184×10^{-17}
100	2.82592×10^{-9}	1.10674×10^{-12}	7.93503×10^{-14}	3.13602×10^{-17}
500	2.26080×10^{-11}	1.86136×10^{-15}	2.53949×10^{-17}	2.29887×10^{-21}
1000	2.82600×10^{-12}	1.17043×10^{-16}	7.93596×10^{-19}	3.64898×10^{-23}
5000	2.26080×10^{-14}	1.88174×10^{-19}	2.53952×10^{-22}	2.36453×10^{-27}
10000	2.82600×10^{-15}	1.17679×10^{-20}	7.93600×10^{-24}	3.70028×10^{-29}

Table 2: Numerically compared errors for the proposed approximations.

to $n!$. Solving explicitly, one finds $n \approx 5.88037$ for eq. (8), and $n \approx 0.311977$, $n \approx 1.21606$, and $n \approx 15.2894$ for eq. (10).

While fig. 2 and table 2 show encouraging results, we will now compare our proposed approximations to known, and oft-cited, approximations. The approximation retained are of comparable computational complexity. We will compare:

- The truncated Stirling series (often ambiguously referred to as "Stirling's Approximation"),
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{11}$$
- Burnside's approximation [2],
$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}, \tag{12}$$
- Gosper's [5],
$$n! \sim \sqrt{\pi}\sqrt{2n+\frac{1}{3}}\left(\frac{n}{e}\right)^n, \tag{13}$$
- Mohanty's and Rummens' [9],
$$n! \sim \sqrt{2\pi}(n+1)^{n+\frac{1}{2}}e^{\frac{1}{12(n+1)}-(n+1)}, \tag{14}$$
- Mortici's [10],
$$n! \sim \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}, \tag{15}$$
- and Nemes' [13],
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n^2-\frac{1}{10}}\right)^n. \tag{16}$$

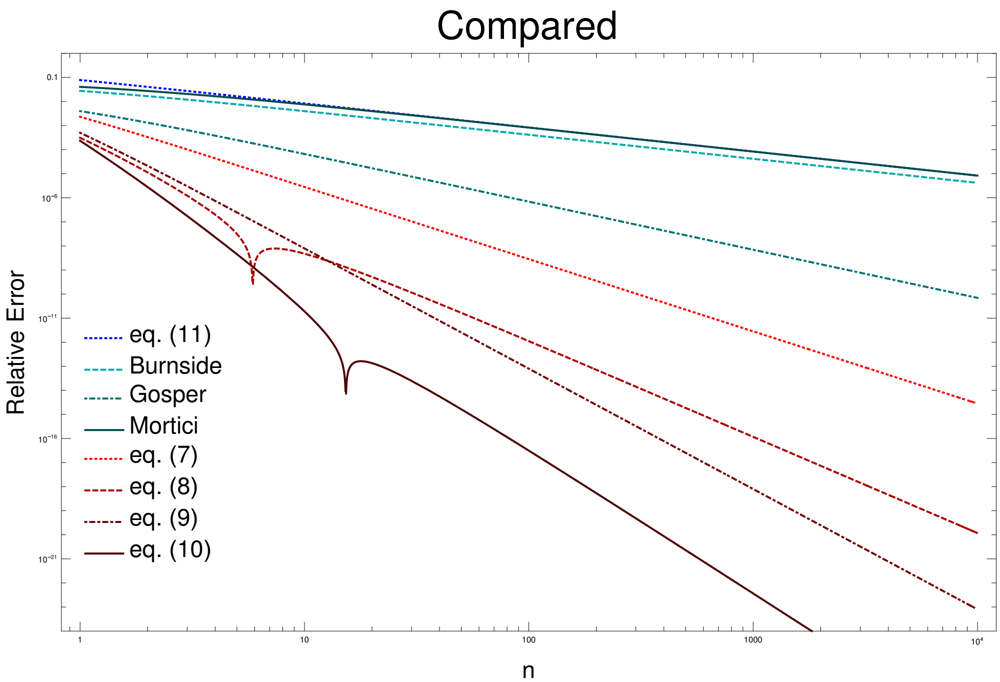

Figure 3: Absolute relative error of proposed approximations compared to other approximations to $n!$.

n	Stirling eq. (11)	Mortici eq. (15)	Burnside eq. (12)	Gosper eq. (13)	M & R eq. (14)	eq. (7)
1	0.07786	0.04050	0.02751	0.00398	0.00033	0.00232284
10	0.00830	0.00755	0.00397	0.00007	2.08209×10^{-6}	2.81813×10^{-6}
20	0.00416	0.00396	0.00203	0.00002	2.99750×10^{-7}	3.53003×10^{-7}
30	0.00277	0.00268	0.00137	7.58471×10^{-6}	9.32145×10^{-8}	1.04634×10^{-7}
40	0.00208	0.00203	0.00103	4.28485×10^{-6}	4.02969×10^{-8}	4.41486×10^{-8}
50	0.00167	0.00163	0.00083	2.74940×10^{-6}	2.09382×10^{-8}	2.26055×10^{-8}
60	0.00139	0.00137	0.00069	1.91259×10^{-6}	1.22370×10^{-8}	1.30823×10^{-8}
70	0.00119	0.00117	0.00059	1.40689×10^{-6}	7.76065×10^{-9}	8.23860×10^{-9}
80	0.00104	0.00103	0.00052	1.07814×10^{-6}	5.22665×10^{-9}	5.51930×10^{-9}
90	0.00093	0.00092	0.00046	8.52471×10^{-7}	3.68603×10^{-9}	3.87641×10^{-9}
100	0.00083	0.00082	0.00041	6.90896×10^{-7}	2.69601×10^{-9}	2.82592×10^{-9}
500	0.00017	0.00017	0.00008	2.77494×10^{-8}	2.20894×10^{-11}	2.26080×10^{-11}
1000	0.00008	0.00008	0.00004	6.94090×10^{-9}	2.76946×10^{-12}	2.82600×10^{-12}
5000	0.00002	0.00002	8.33253×10^{-6}	2.77749×10^{-10}	2.22089×10^{-14}	2.26080×10^{-14}
10000	8.33330×10^{-6}	8.33247×10^{-6}	4.16647×10^{-6}	6.94409×10^{-11}	2.77694×10^{-15}	2.82600×10^{-15}

Table 3: All approximations compared, absolute relative error, first part.

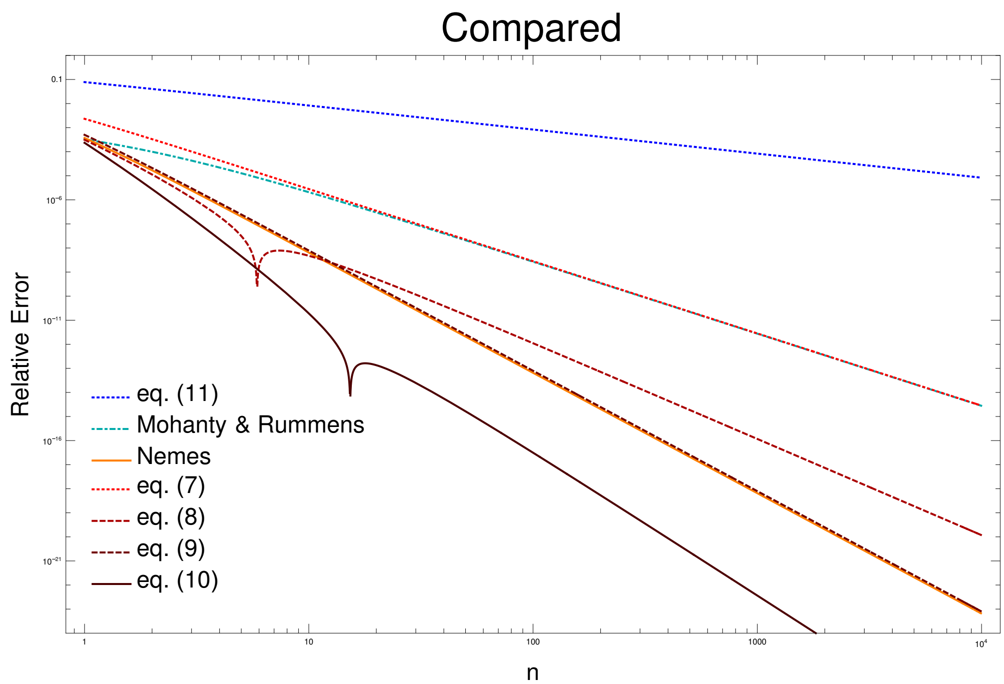

Figure 4: Absolute relative error of proposed approximations compared to other approximations to $n!$.

Fig. 3 and table 3 present the relative errors of our proposed approximation, eq. (7), against those from Burnside, Mortici, and Gosper. From table 3, we see that Mortici's approximation, eq. (15), does marginally better than Stirling's truncated series, eq. (11). Burnside's approximation has a relative error approximately half of the truncated Stirling series, eq. (11). Compared to the other results, one can see that Burnside's approximation is of historical interest at best. Gosper's approximation fares much better, but still worse than Mohanty's and Rummens'. Mohanty's and Rummens' approximation does better than eq. (7), but just so.

Fig. 4 presents the relative absolute errors of our proposed approximations against Nemes', eq. (16), and Mohanty's and Rummens', eq. (14). Note that Nemes' approximation and eq. (9) overlap on the figure, but Nemes' does better as detailed in table 4. Table 4 presents the same results, but this time including Stirling's series, eq. (1), with its first ten terms. Nemes' approximation does better than eq. (9), but eq. (10) does much better than either, while the ten term Stirling series shows errors tens of orders of magnitude smaller.

Another interesting measure of accuracy is the number of correct leading digits, shown in table 5. Indeed, an approximation is "perfect" if it gives as many correct digits as the number representation is capable of storing, since any additional correct digits will be lost to quantization. If one uses single-precision IEEE 754 floating point numbers, one expects about 7 significant digits, because its mantissa is 24 bits long (of which only 23 are explicitly stored [21]), and we have $\log_{10} 2^{24} \approx 7.22$. With double-precision floating point, the mantissa is 53 bits long, and we expect $\log_{10} 2^{53} \approx 15.95$, or about 16, significant digits. At 7 significant digits, eqs. (8) to (10), as well as eq. (16), are basically equivalent in terms of accuracy. With 16 digits, only eq. (10) is cromulent.

n	eq. (7)	eq. (8)	eq. (9)	eq. (10)	Nemes eq. (16)	Stirling eq. (1)*
1	0.00232284	0.000311662	0.000501953	0.000234244	0.000372486	0.000467142
10	2.81813×10^{-6}	4.75012×10^{-9}	7.87352×10^{-9}	1.91489×10^{-11}	6.47042×10^{-9}	1.13891×10^{-14}
20	3.53003×10^{-7}	5.15163×10^{-10}	2.47478×10^{-10}	1.35261×10^{-13}	2.03553×10^{-10}	2.26443×10^{-18}
30	1.04634×10^{-7}	1.16269×10^{-10}	3.26261×10^{-11}	2.48111×10^{-14}	2.68387×10^{-11}	1.40009×10^{-20}
40	4.41486×10^{-8}	3.90880×10^{-11}	7.74545×10^{-12}	5.57148×10^{-15}	6.37174×10^{-12}	2.30179×10^{-21}
50	2.26055×10^{-8}	1.65760×10^{-11}	2.53852×10^{-12}	1.64257×10^{-15}	2.08831×10^{-12}	3.44988×10^{-22}
60	1.30823×10^{-8}	8.17576×10^{-12}	1.02028×10^{-12}	5.90763×10^{-16}	8.39339×10^{-13}	6.62692×10^{-23}
70	8.23860×10^{-9}	4.48322×10^{-12}	4.72080×10^{-13}	2.45804×10^{-16}	3.88358×10^{-13}	1.58004×10^{-23}
80	5.51930×10^{-9}	2.65882×10^{-12}	2.42145×10^{-13}	1.14196×10^{-16}	1.99201×10^{-13}	4.47552×10^{-24}
90	3.87641×10^{-9}	1.67487×10^{-12}	1.34377×10^{-13}	5.78184×10^{-17}	1.10546×10^{-13}	1.45461×10^{-24}
100	2.82592×10^{-9}	1.10674×10^{-12}	7.93503×10^{-14}	3.13602×10^{-17}	6.52774×10^{-14}	5.28507×10^{-25}
500	2.26080×10^{-11}	1.86136×10^{-15}	2.53949×10^{-17}	2.29887×10^{-21}	2.08906×10^{-17}	6.98451×10^{-32}
1000	2.82600×10^{-12}	1.17043×10^{-16}	7.93596×10^{-19}	3.64898×10^{-23}	6.52832×10^{-19}	7.01285×10^{-35}
5000	2.26080×10^{-14}	1.88174×10^{-19}	2.53952×10^{-22}	2.36453×10^{-27}	2.08907×10^{-22}	7.33848×10^{-42}
10000	2.82600×10^{-15}	1.17679×10^{-20}	7.93600×10^{-24}	3.70028×10^{-29}	6.52833×10^{-24}	7.18569×10^{-45}

Table 4: All approximations compared, absolute relative error, second part. *With the first ten terms.

n	$n!$	Stirling eq. (11)	Gosper eq. (13)	M & R eq. (14)	eq. (7)	eq. (8)	eq. (9)	eq. (10)	Nemes eq. (16)	Stirling eq. (1)*
1	1	0	0	0	0	0	0	0	0	0
10	7	1	4	6	5	6	6	6	6	6
20	19	2	4	7	7	9	9	13	9	18
30	33	2	5	7	7	10	10	13	10	20
40	48	2	5	6	6	10	11	14	11	20
50	65	2	6	8	8	10	11	15	11	21
60	82	2	5	7	7	10	11	14	11	21
70	101	3	6	8	8	9	13	15	13	23
80	119	2	5	8	8	11	12	14	12	23
90	139	3	6	9	8	12	13	17	13	24
100	158	2	5	8	8	10	12	15	12	23
500	1135	2	7	9	9	15	17	19	17	30
1000	2568	4	7	11	11	15	18	21	18	34
5000	16326	5	9	13	13	19	19	25	21	41
10000	35660	5	10	14	14	20	22	28	22	43

Table 5: Number of correct leading digits for different approximations. The column with $n!$ shows the number of digits in $n!$. *With the first 10 terms.

5 Implementation and Computational Complexity

To be interesting, an approximation formula must, in addition of being accurate, be easily and efficiently computed. Some of the approximations we used for comparison raise the correction term to the n-th (or $(n+\frac{1}{2})$-th) power, possibly requiring both raising to the n-th power and extracting a square root. If we have $n \in \mathbb{N}$, we can evaluate the exponentiation in $O(\log n)$ steps using the successive squaring method (an old idea, see [3, p. 76] and [15]), but this seems too restrictive, as we will want $n \in \mathbb{R}$. Rational functions of degree d, on the other hand, can be evaluated in $O(d)$ products, whether n is a natural or a real number, and independently of its magnitude—although it will become costlier when n grows if exact arithmetic is used.

Eqs. (7) to (10) are best evaluated using reduced rational functions in order to minimize the number of multiplications and divisions required. The correction term of eq. (7) can be rewritten in many ways,

$$1 + \frac{1}{12n - \frac{1}{2}} = \frac{12n + \frac{1}{2}}{12n - \frac{1}{2}} = \frac{24n + 1}{24n - 1} = 1 + \frac{2}{24n - 1}. \tag{17}$$

Eq. (8) is probably best expressed as eq. (8b), resulting in three products and one division, while eq. (8a) needs two products and two divisions, a gain if we consider the divisions to be much more expensive than products. For eq. (9), we find that, expressed as eq. (9a), the correction term requires five products and two divisions. Expressed as eq. (9b), we notice that if we let $t = n^2$, we can rewrite the correction term as

$$\frac{17280t + 720n - 556}{207360tn + 571},$$

which now requires the same number of products, but with only one division. If we apply the same type of simplification to eq. (10b), again with $t = n^2$, we find that

$$\frac{1451520n^3 + 60480n^2 - 46704n - 3997}{17418240n^4 - 163879} = \frac{(1451520n + 60480)t - 46704n - 3997}{17418240t^2 - 163879}$$

can now be evaluated with six products and one division instead of ten products and two divisions in its original form, eq. (10a). As the degree d grows, the resulting rational functions may be subject to other optimization strategies [1, 12], but the evaluation will require, at worse, $O(d)$ multiplications, if only by using Horner's method for evaluating polynomials [7, 14].

6 Conclusion

Hodgman's correction to eq. (2), eq. (3), lead us to the observation that since these functions can be seen as two special cases of a continuous function, by the intermediate value theorem, there must be a correction for which the error is zero, that is, there must exist a function $\alpha(d, n)$ in eq. (4) that yields $n!$ exactly. From this observation, we proposed a novel family of approximations, each approximation being the optimal rational function $\alpha(d, n)$ of degree d

in n. We then solved exactly for a few special cases, first for $\alpha(0, n)$, a constant, then for small degrees $1 \leqslant d \leqslant 3$. In doing so, we showed that $\alpha = -\frac{1}{2}$ is the optimal constant correction. We also showed the rational functions of small degrees fare quite well compared to oft-cited approximations for $n!$. We remarked that if the special case $\alpha(2, n)$ is quite comparable to Nemes' approximation, with the latter being a still better; $\alpha(3, n)$ is at least three orders of magnitude better than any of the other approximation used for comparison. Lastly, we discussed computational complexity for the correction term, remarking that our solutions based on rational functions of (small) degree d will require at most $O(d)$ products and one division, while many of the other approximations will require $O(\log n)$ products, or require exponentiation by an arbitrary number, making our proposal also attractive computation-wise.

References

[1] E. G. Belaga. On Computing Polynomials in One Variable with Initial Preconditioning of the Coefficients. *Problemi Kibernetiki*, 5:7–15, 1961. In Russian, with original title: *О вычислений значений многочленов от одного переменного с предварительной обработкой коэффициентов*.

[2] William Burnside. A Rapidly Convergent Series for $\log N!$. *Messenger of Mathematics*, 46:157–159, 1917.

[3] Bibhutibhusan Datta and Avadhesh Narayan Singh. *History of Hindu Mathematics: A Source Book, Part I.* Motilal Barnasi Das, 1935.

[4] Steven A. Feller and Joseph E. Kasper. How Good are the Common Approximations used in Physics? *American Journal of Physics*, 50(8):682–683, 1982.

[5] R. William Gosper, Jr. Decision Procedure for Indefinite Hypergeometric Summation. *Procs. Nat. Acad. Science USA*, 75(1):40–42, 1978.

[6] Charles D. Hodgman, editor. *CRC Standard Mathematical Tables.* Chemical Rubber Publishing Company, 10th edition, 1955.

[7] W. G. Horner. A New Method of Solving Numerical Equations of All Orders, by Continuous Approximations. *Philosophical Transactions of the Royal Society of London*, 109:308–335, 1819.

[8] Dawei Lu. A New Sharp Approximation for the Gamma Function Related to Burnside's Formula. *The Ramanujan Journal*, 35(1):121–129, 2014.

[9] S. Mohanty and Frans H. A. Rummens. Comment on: "An Improved Analytical Approximation to $n!$". *J. of Chemical Physics*, 80(1):591, 1984.

[10] Cristinel Mortici. On Gosper's Formula to the Gamma Function. *J. of Mathematical Inequalities*, 5(4):611–614, 2011.

[11] Cristinel Mortici. A Continued Fraction Approximation of the Gamma Function. *J. of Mathematical Analysis and Applications*, 402(2):405–410, 2013.

[12] Ian Munro and Michael Paterson. Optimal Algorithms for Parallel Polynomial Evaluation. *J. of Computer and System Sciences*, 7(2):189–198, April 1973.

[13] Gergő Nemes. New Asymptotic Expansion for the Gamma Function. *Archiv der Mathematik*, 95:161–169, June 2010.

[14] V. Ya. Pan. Methods of Computing Values of Polynomials. *Russian Mathematical Surveys*, 21:105–136, 1966.

[15] A. S. Saidan. *The Arithmetic of Al-Uqlīdisī: The Story of Hindu-Arabic Arithmetic as told in the Kitāb al-Fuṣū fī al-Ḥisāb al-Hindī*. D. Reidel Publishing Co., 1978.

[16] Neil J. A. Sloane. A001163: Stirling's formula: numerators of asymptotic series for Gamma function. http://oeis.org/A001163.

[17] Neil J. A. Sloane. A001164: Stirling's formula: denominators of asymptotic series for Gamma function. http://oeis.org/A001164.

[18] James Stirling. *Methodus Differentialis, sive Tractatus de Summatione et Interpolatione Serierum Infinitarium*. G. Strahan, London, 1730.

[19] Yitzhak Weissman. An Improved Analytical Approximation to $n!$. *American Journal of Physics*, 51(9):778–779, 1983.

[20] Zhen-Hang Yang and Jing-Feng Tian. An Accurate Approximation Formula for Gamma Function. *J. of Inequalities and Applications*, pages 2018:56–2018:63, 2018.

[21] Dan Zuras and Mike Cowlishaw, editors. *IEEE 754-2008 – Standard for Floating Point Arithmetic*. The Institute of Electrical and Electronics Engineers (IEEE), 2008.